

# ON COMPACTNESS OF HANKEL AND THE $\bar{\partial}$ -NEUMANN OPERATORS ON HARTOGS DOMAINS IN $\mathbb{C}^2$

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**ABSTRACT.** We prove that on smooth bounded pseudoconvex Hartogs domains in  $\mathbb{C}^2$  compactness of the  $\bar{\partial}$ -Neumann operator is equivalent to compactness of all Hankel operators with symbols smooth on the closure of the domain.

Let  $\Omega$  be a bounded pseudoconvex domain in  $\mathbb{C}^n$  and  $L^2_{(0,q)}(\Omega)$  denote the space of square integrable  $(0,q)$  forms for  $0 \leq q \leq n$ . The complex Laplacian  $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$  is a densely defined, closed, self-adjoint linear operator on  $L^2_{(0,q)}(\Omega)$ . Hörmander in [Hör65] showed that when  $\Omega$  is bounded and pseudoconvex,  $\square$  has a bounded solution operator  $N_q$ , called the  $\bar{\partial}$ -Neumann operator for all  $q$ . Kohn in [Koh63] showed that the Bergman projection, denoted by  $\mathbf{B}$  below, is connected to the  $\bar{\partial}$ -Neumann operator via the following formula

$$\mathbf{B} = \mathbf{I} - \bar{\partial}^* N_1 \bar{\partial}$$

where  $\mathbf{I}$  denotes the identity operator. For more information about the  $\bar{\partial}$ -Neumann problem we refer the reader to two books [CS01, Str10].

Let  $A^2(\Omega)$  denote the space of square integrable holomorphic functions on  $\Omega$  and  $\phi \in L^\infty(\Omega)$ . The Hankel operator with symbol  $\phi$ ,  $H_\phi : A^2(\Omega) \rightarrow L^2(\Omega)$  is defined by

$$H_\phi g = [\phi, \mathbf{B}]g = (\mathbf{I} - \mathbf{B})(\phi g).$$

Using Kohn's formula one can immediately see that

$$H_\phi g = \bar{\partial}^* N_1(g\bar{\partial}\phi)$$

for  $\phi \in C^1(\bar{\Omega})$ . It is clear that  $H_\phi$  is a bounded operator; however, its compactness depends on both the function theoretic properties of the symbol  $\phi$  as well as the geometry of the boundary of the domain  $\Omega$  (see [ÇŞ09]).

The following observation is relevant to our work here. Let  $\Omega$  be a bounded pseudoconvex domain in  $\mathbb{C}^n$  and  $\phi \in C(\bar{\Omega})$ . If  $\bar{\partial}$ -Neumann operator  $N_1$  is compact on  $L^2_{(0,1)}(\Omega)$  then the Hankel operator  $H_\phi$  is compact (see [Str10, Proposition 4.1]).

We are interested in the converse of this observation. Namely,

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2010 *Mathematics Subject Classification.* Primary 32W05; Secondary 47B35.

*Key words and phrases.* Hankel operators,  $\bar{\partial}$ -Neumann problem, Hartogs domains.

The work of the second author was partially supported by a grant from the Simons Foundation (#353525), and also by a University of Michigan-Dearborn CASL Faculty Summer Research Grant. Both authors would like to thank the American Institute of Mathematics for hospitality for hosting a workshop during which this project was started.

Assume that  $\Omega$  is a bounded pseudoconvex domain in  $\mathbb{C}^n$  and  $H_\phi$  is compact on  $A^2(\Omega)$  for all symbols  $\phi \in C(\overline{\Omega})$ . Then is the  $\bar{\partial}$ -Neumann operator  $N_1$  compact on  $L^2_{(0,1)}(\Omega)$ ?

This is known as D'Angelo's question and has first appeared in [FS01, Remark 2].

The answer to D'Angelo's question is still open in general but there are some partial results. Fu and Straube in [FS98] showed that the answer is yes if  $\Omega$  is convex. Çelik and the first author [ÇŞ12, Corollary 1] observed that if  $\Omega$  is not pseudoconvex then the answer to D'Angelo's question may be no. Indeed, they constructed an annulus type domain  $\Omega$  where  $H_\phi$  is compact on  $A^2(\Omega)$  for all symbols  $\phi \in C(\overline{\Omega})$ ; yet, the  $\bar{\partial}$ -Neumann operator  $N_1$  is not compact on  $L^2_{(0,1)}(\Omega)$ .

*Remark 1.* One can extend the definition of Hankel operators from holomorphic functions to the  $\bar{\partial}$ -closed  $(0, q)$  forms (denoted by  $K^2_{(0,q)}(\Omega)$ ) and ask the analogous problem at the forms level. In this case, an affirmative answer was obtained in [ÇŞ14]. Namely, for  $1 \leq q \leq n-1$  if  $H^q_\phi = [\phi, \mathbf{B}_q]$  is compact on  $K^2_{(0,q)}(\Omega)$  for all symbols  $\phi \in C^\infty(\overline{\Omega})$  then the  $\bar{\partial}$ -Neumann operator  $N_{q+1}$  is compact on  $L^2_{(0,q)}(\Omega)$ .

In this paper, we provide an affirmative answer to D'Angelo's question on smooth bounded pseudoconvex Hartogs domains in  $\mathbb{C}^2$ .

**Theorem 1.** *Let  $\Omega$  be a smooth bounded pseudoconvex Hartogs domain in  $\mathbb{C}^2$ . The  $\bar{\partial}$ -Neumann operator  $N_1$  is compact on  $L^2_{(0,1)}(\Omega)$  if and only if  $H_\psi$  is compact on  $A^2(\Omega)$  for all  $\psi \in C^\infty(\overline{\Omega})$ .*

As mentioned above, compactness of  $N_1$  implies that  $H_\psi$  is compact on any bounded pseudoconvex domain (see [FS01, Proposition 4] or [Str10, Proposition 4.1]). The key ingredient of our proof of the converse is the characterization of the compactness of  $N_1$  in terms of ground state energies of certain Schrödinger operators as previously explored in [FS02, CF05].

We will need few lemmas before we prove Theorem 1.

**Lemma 1.** *Let  $A(a, b) = \{z \in \mathbb{C} : a < |z| < b\}$  for  $0 < a < b < \infty$  and  $d_{ab}(w)$  be the distance from  $w$  to the boundary of  $A(a, b)$ . Then there exists  $C > 0$  such that*

$$\int_{A(a,b)} (d_{ab}(w))^2 |w|^{2n} dV(w) \leq \frac{C}{n^2} \int_{A(a,b)} |w|^{2n} dV(w)$$

for nonzero integer  $n$ .

*Proof.* We will use the fact that  $d_{ab}(w) = \min\{b - |w|, |w| - a\}$  with polar coordinates to compute the first integral. One can compute that

$$\int_{A(a,b)} |w|^{2n} dV(w) = \frac{\pi}{n+1} (b^{2n+2} - a^{2n+2})$$

for  $n \neq -1$ . Let  $c = \frac{a+b}{2}$ . Then

$$\begin{aligned}
\int_{A(a,b)} (d_{ab}(w))^2 |w|^{2n} dV(w) &= \int_{A(a,c)} (|w| - a)^2 |w|^{2n} dV(w) \\
&\quad + \int_{A(c,b)} (b - |w|)^2 |w|^{2n} dV(w) \\
&= 2\pi \int_a^c (a^2 \rho^{2n+1} - 2a\rho^{2n+2} + \rho^{2n+3}) d\rho \\
&\quad + 2\pi \int_c^b (b^2 \rho^{2n+1} - 2b\rho^{2n+2} + \rho^{2n+3}) d\rho \\
&= 2\pi (b^{2n+4} - a^{2n+4}) \left( \frac{1}{2n+2} - \frac{2}{2n+3} + \frac{1}{2n+4} \right) \\
&\quad + 2\pi (a^2 - b^2) \frac{c^{2n+2}}{2n+2} + 4\pi (b-a) \frac{c^{2n+3}}{2n+3} \\
&= \frac{\pi (b^{2n+4} - a^{2n+4})}{(n+1)(n+2)(2n+3)} - \frac{\pi c^{2n+2} (b^2 - a^2)}{(n+1)(2n+3)}.
\end{aligned}$$

In the last equality we used the fact that  $c = \frac{a+b}{2}$ . Then one can show that

$$\lim_{n \rightarrow \pm\infty} \frac{n^2 \int_{A(a,b)} (d_{ab}(w))^2 |w|^{2n} dV(w)}{\int_{A(a,b)} |w|^{2n} dV(w)} = \frac{b^2}{2}.$$

Therefore, there exists  $C > 0$  such that

$$\int_{A(a,b)} (d_{ab}(w))^2 |w|^{2n} dV(w) \leq \frac{C}{n^2} \int_{A(a,b)} |w|^{2n} dV(w)$$

for nonzero integer  $n$ . □

We note that throughout the paper  $\|\cdot\|_{-1}$  denotes the Sobolev  $-1$  norm.

**Lemma 2.** Let  $\Omega = \{(z, w) \in \mathbb{C}^2 : z \in D \text{ and } \phi_1(z) < |w| < \phi_2(z)\}$  be a bounded Hartogs domain. Then there exists  $C > 0$  such that

$$\|g(z)w^n\|_{-1} \leq \frac{C}{n} \|g(z)w^n\|$$

for any  $g \in L^2(D)$  and nonzero integer  $n$ , as long as the right hand side is finite.

*Proof.* We will denote the distance from  $(z, w)$  to the boundary of  $\Omega$  by  $d_\Omega(z, w)$ . We note that  $W^{-1}(\Omega)$  is the dual of  $W_0^1(\Omega)$ , the closure of  $C_0^\infty(\Omega)$  in  $W^1(\Omega)$ . Furthermore,

$$\|f\|_{-1} = \sup\{|\langle f, \phi \rangle| : \phi \in C_0^\infty(\Omega), \|\phi\|_1 \leq 1\}$$

for  $f \in W^{-1}(\Omega)$ . Then there exists  $C_1 > 0$  such that

$$\|f\|_{-1} \leq \|d_\Omega f\| \sup\{\|\phi/d_\Omega\| : \phi \in C_0^\infty(\Omega), \|\phi\|_1 \leq 1\} \leq C_1 \|d_\Omega f\|.$$

In the second inequality above we used the fact that (see [CS01, Proof of Theorem C.3]) there exists  $C_1 > 0$  such that  $\|\phi/d_\Omega\| \leq C_1 \|\phi\|_1$  for all  $\phi \in W_0^1(\Omega)$ .

Let  $d_z(w)$  denote the distance from  $w$  to the boundary of  $A(\phi_1(z), \phi_2(z))$ . Then there exists  $C_1 > 0$  such that

$$\begin{aligned} \|g(z)w^n\|_{-1}^2 &\leq C_1 \int_{\Omega} (d_{\Omega}(z, w))^2 |g(z)|^2 |w|^{2n} dV(z, w) \\ &\leq C_1 \int_D |g(z)|^2 \int_{\phi_1(z) < |w| < \phi_2(z)} (d_z(w))^2 |w|^{2n} dV(w). \end{aligned}$$

Lemma 1 and the assumption that  $\Omega$  is bounded imply that there exists  $C_2 > 0$  such that

$$\int_{\phi_1(z) < |w| < \phi_2(z)} (d_z(w))^2 |w|^{2n} dV(w) \leq \frac{C_2}{n^2} \int_{\phi_1(z) < |w| < \phi_2(z)} |w|^{2n} dV(w).$$

Then

$$\begin{aligned} \int_D |g(z)|^2 \int_{\phi_1(z) < |w| < \phi_2(z)} (d_z(w))^2 |w|^{2n} dV(w) &\leq \frac{C_2}{n^2} \int_D |g(z)|^2 \int_{\phi_1(z) < |w| < \phi_2(z)} |w|^{2n} dV(w) \\ &= \frac{C_2}{n^2} \|g(z)w^n\|^2. \end{aligned}$$

Therefore, for  $C = \sqrt{C_1 C_2}$  we have  $\|g(z)w^n\|_{-1} \leq \frac{C}{n} \|g(z)w^n\|$  for nonzero integer  $n$ .  $\square$

**Lemma 3.** *Let  $\Omega$  be a bounded pseudoconvex domain in  $\mathbb{C}^n$  and  $\psi \in C^1(\overline{\Omega})$ . Then  $H_{\psi}$  is compact if and only if for any  $\varepsilon > 0$  there exists  $C_{\varepsilon} > 0$  such that*

$$(1) \quad \|H_{\psi}h\|^2 \leq \varepsilon \|h\bar{\partial}\psi\| \|h\| + C_{\varepsilon} \|h\bar{\partial}\psi\|_{-1} \|h\|$$

for  $h \in A^2(\Omega)$ .

*Proof.* First assume that  $H_{\psi}$  is compact. Then

$$\|H_{\psi}h\|^2 = \langle H_{\psi}^* H_{\psi} h, h \rangle \leq \|H_{\psi}^* H_{\psi} h\| \|h\|$$

for  $h \in A^2(\Omega)$ . Compactness of  $H_{\psi}$  implies that  $H_{\psi}^*$  is compact. Now we apply the compactness estimate in [D'A02, Proposition V.2.3] to  $H_{\psi}^*$ . For  $\varepsilon > 0$  there exists a compact operator  $K_{\varepsilon}$  such that

$$\begin{aligned} \|H_{\psi}^* H_{\psi} h\| &\leq \frac{\varepsilon}{2\|\bar{\partial}^* N\|} \|H_{\psi} h\| + \|K_{\varepsilon} H_{\psi} h\| \\ &\leq \frac{\varepsilon}{2} \|h\bar{\partial}\psi\| + \|K_{\varepsilon} H_{\psi} h\|. \end{aligned}$$

In the second inequality we used the fact that  $H_{\psi}h = \bar{\partial}^* N(h\bar{\partial}\psi)$ . Since  $\Omega$  is bounded pseudoconvex  $\bar{\partial}^* N$  is bounded and hence  $K_{\varepsilon} \bar{\partial}^* N$  is compact. Now we use the fact that  $H_{\psi}h = \bar{\partial}^* N(h\bar{\partial}\psi)$  and [Str10, Lemma 4.3] for the compact operator  $K_{\varepsilon} \bar{\partial}^* N$  to conclude that there exists  $C_{\varepsilon} > 0$  such that

$$\|K_{\varepsilon} H_{\psi} h\| \leq \frac{\varepsilon}{2} \|h\bar{\partial}\psi\| + C_{\varepsilon} \|h\bar{\partial}\psi\|_{-1}.$$

Therefore, for  $\varepsilon > 0$  there exists  $C_{\varepsilon} > 0$  such that

$$\|H_{\psi}h\|^2 \leq \varepsilon \|h\bar{\partial}\psi\| \|h\| + C_{\varepsilon} \|h\bar{\partial}\psi\|_{-1} \|h\|$$

for  $h \in A^2(\Omega)$ .

To prove the converse assume (1) and choose  $\{h_j\}$  a sequence in  $A^2(\Omega)$  such that  $\{h_j\}$  converges to zero weakly. Then the sequence  $\{h_j\}$  is bounded and  $\|h_j \bar{\partial} \psi\|_{-1}$  converges to 0 (as the imbedding from  $L^2$  into Sobolev  $-1$  is compact). The inequality (1) implies that there exists  $C > 0$  such that for every  $\varepsilon > 0$  there exists  $J$  such that  $\|H_\psi h_j\|^2 \leq C\varepsilon$  for  $j \geq J$ . That is,  $\{H_\psi h_j\}$  converges to 0. That is,  $H_\psi$  is compact.  $\square$

The following lemma is contained in [Sah12, Remark 1]. The superscripts on the Hankel operators are used to emphasize the domains.

**Lemma 4** ([Sah12]). *Let  $\Omega_1$  be a bounded pseudoconvex domain in  $\mathbb{C}^n$  and  $\Omega_2$  be a bounded strongly pseudoconvex domain in  $\mathbb{C}^n$  with  $C^2$ -smooth boundary. Assume that  $U = \Omega_1 \cap \Omega_2$  is connected,  $\phi \in C^1(\bar{\Omega}_1)$ , and  $H_\phi^{\Omega_1}$  is compact on  $A^2(\Omega_1)$ . Then  $H_\phi^U$  is compact on  $A^2(U)$ .*

Now we are ready to prove Theorem 1.

*Proof of Theorem 1.* We present proof of the nontrivial direction. That is, we assume that  $H_\psi$  is compact on  $A^2(\Omega)$  for all  $\psi \in C^\infty(\bar{\Omega})$  and prove that  $N_1$  is compact. Our proof is along the lines of the proof of [CF05, Theorem 1.1].

Let  $\rho(z, w)$  be a smooth defining function for  $\Omega$  that is invariant under rotations in  $w$ . That is,  $\rho(z, w) = \rho(z, |w|)$ ,

$$\Omega = \{(z, w) \in \mathbb{C}^2 : \rho(z, w) < 0\},$$

and  $\nabla \rho$  is nonvanishing on  $b\Omega$ . Let  $\Gamma_0 = \{(z, w) \in b\Omega : \rho_{|w|}(z, |w|) = 0\}$  and

$$\Gamma_k = \{(z, w) \in b\Omega : |\rho_{|w|}(z, |w|)| \geq 1/k\}$$

for  $k = 1, 2, \dots$ . We will show that  $\Gamma_k$  is  $B$ -regular for  $k = 0, 1, 2, \dots$  by establishing the estimates (2) and (3) below and invoking [CF05, Lemma 10.2]. Then

$$b\Omega = \bigcup_{k=0}^{\infty} \Gamma_k$$

and [Sib87, Proposition 1.9] implies that  $b\Omega$  is  $B$ -regular (satisfies Property (P) in Catlin's terminology). This will be enough to conclude that  $N_1$  is compact on  $L_{(0,1)}^2(\Omega)$ .

The proof of the fact that  $\Gamma_0$  is  $B$ -regular is essentially contained in [CF05, Lemma 10.1] together with the following fact: Let  $\Omega$  be a smooth bounded pseudoconvex domain in  $\mathbb{C}^2$ . If  $H_{\bar{z}}$  and  $H_{\bar{w}}$  are compact on  $A^2(\Omega)$  then there is no analytic disc in  $b\Omega$  (see [ČŠ09, Corollary 1]).

Now we will prove that  $\Gamma_k$  is  $B$ -regular for any fixed  $k \geq 1$ . Let  $(z_0, w_0) \in \Gamma_k$ , we argue in two cases. The first case is when  $\rho_{|w|}(z_0, |w_0|) < 0$  and the second case is  $\rho_{|w|}(z_0, |w_0|) > 0$ .

We continue with the first case. Assume that  $b\Omega$  near  $(z_0, w_0)$  is given by  $|w| = e^{-\varphi(z)}$ . Let  $D(z_0, r)$  denote the disc centered at  $z_0$  with radius  $r$  and

$$U_{a,b} = D(z_0, a) \times \{w \in \mathbb{C} : |w_0| - b < |w| < |w_0| + b\}$$

for  $a, b > 0$ . Then let us choose  $a, a_1, b, b_1 > 0$  such that  $a_1 > a, b_1 > |w_0| + b$ , the open sets

$$U = \Omega \cap U_{a,b} = \left\{ (z, w) \in \mathbb{C}^2 : z \in D(z_0, a), e^{-\varphi(z)} < |w| < |w_0| + b \right\}$$

and  $U_1 = \Omega \cap U^{a_1, b_1}$  are connected where

$$U^{a_1, b_1} = \left\{ (z, w) \in \mathbb{C}^2 : \frac{|z - z_0|^2}{a_1^2} + \frac{|w|^2}{b_1^2} < 1 \right\},$$

and finally  $\overline{U} \subset U_1$ . Then

$$U_1 = \left\{ (z, w) \in \mathbb{C}^2 : z \in V_1, e^{-\varphi(z)} < |w| < e^{-\alpha(z)} \right\}$$

where  $V_1$  is a domain in  $\mathbb{C}$  such that  $\overline{D(z_0, a)} \subset V_1 \subset D(z_0, a_1)$  and

$$\alpha(z) = \log a_1 - \log b_1 - \frac{1}{2} \log(a_1^2 - |z - z_0|^2).$$

One can check that  $\alpha$  is subharmonic on  $D(z_0, a_1)$ , while pseudoconvexity of  $\Omega$  implies that the function  $\varphi$  is superharmonic on  $D(z_0, a_1)$ . Furthermore, since  $B$ -regularity is invariant under holomorphic change of coordinates, by mapping under  $(z, w) \rightarrow (z, \lambda w)$  for some  $\lambda > 1$ , we may assume that

$$U_1 \subset D(z_0, a_1) \times \{w \in \mathbb{C} : |w| > 1\}.$$

For any  $\beta \in C_0^\infty(D(z_0, a))$  let us choose  $\psi \in C^\infty(\overline{V_1})$  such that  $\psi_{\overline{z}} = \beta$ . Lemma 4 implies that the Hankel operator  $H_\psi^{U_1}$  (we use the superscript  $U_1$  to emphasize the domain) is compact on the Bergman space  $A^2(U_1)$ .

Let

$$\lambda_n(z) = -\log \left( \frac{\pi}{n-1} \left( e^{(2n-2)\varphi(z)} - e^{(2n-2)\alpha(z)} \right) \right)$$

for  $n = 2, 3, \dots$ . One can check that since  $\varphi$  is superharmonic and  $\alpha$  is subharmonic, the function  $\lambda_n$  is subharmonic. Let  $S_{\lambda_n}^{V_1}$  be the canonical solution operator for  $\bar{\partial}$  on  $L^2(V_1, \lambda_n)$ . If  $f_n = H_\psi^{U_1} w^{-n}$  then we claim that

$$f_n(z, w) = g_n(z) w^{-n}$$

where  $g_n = S_{\lambda_n}^{V_1}(\beta d\bar{z})$  and  $n = 2, 3, \dots$ . Clearly  $H_\psi^{U_1} w^{-n} = f_n \in L^2(U_1)$  and

$$\bar{\partial} g_n(z) w^{-n} = \beta(z) w^{-n} d\bar{z}.$$

To prove the claim we will just need to show that  $g_n(z) w^{-n}$  is orthogonal to  $A^2(U_1)$ . That is, we need to show that  $\langle g_n(z) w^{-n}, h(z) w^m \rangle_{U_1} = 0$  for any  $h(z) \in A^2(V_1)$  and  $m \in \mathbb{Z}$ . Then

$$\begin{aligned} \langle g_n(z) w^{-n}, h(z) w^m \rangle_{U_1} &= \int_{U_1} g_n(z) w^{-n} \overline{h(z) w^m} dV(z) dV(w) \\ &= \int_{V_1} g_n(z) \overline{h(z)} dV(z) \int_{e^{-\varphi(z)} < |w| < e^{-\alpha(z)}} w^{-n} \overline{w^m} dV(w). \end{aligned}$$

Unless  $m = -n$  the integral  $\int_{e^{-\varphi(z)} < |w| < e^{-\alpha(z)}} w^{-n} \overline{w^m} dV(w) = 0$ . So let us assume that  $m = -n$ . In that case we get

$$\int_{V_1} g_n(z) \overline{h(z)} dV(z) \int_{e^{-\varphi(z)} < |w| < e^{-\alpha(z)}} w^{-n} \overline{w^m} dV(w) = \int_{V_1} g_n(z) \overline{h(z)} e^{-\lambda_n(z)} dV(z).$$

The integral on the right hand side above is zero because  $g_n$  is orthogonal to  $A^2(V_1, \lambda_n)$ . Therefore,

$$g_n(z) w^{-n} = H_\psi^{U_1} w^{-n}.$$

The equality above implies that  $\frac{\partial g_n}{\partial \bar{z}} = \frac{\partial \psi}{\partial \bar{z}} = \beta$ . Then the compactness estimate (1) implies that

$$\begin{aligned} \int_{D(z_0, a)} |g_n(z)|^2 e^{-\lambda_n(z)} dV(z) &\leq \|g_n(z) w^{-n}\|_{U_1}^2 \\ &\leq \varepsilon \|\beta(z) w^{-n}\|_{U_1} \|w^{-n}\|_{U_1} + C_\varepsilon \|\beta(z) w^{-n}\|_{W^{-1}(U_1)} \|w^{-n}\|_{U_1} \\ &= \varepsilon \left( \int_{D(z_0, a)} |\beta(z)|^2 e^{-\lambda_n(z)} dV(z) \right)^{1/2} \left( \int_{V_1} e^{-\lambda_n(z)} dV(z) \right)^{1/2} \\ &\quad + C_\varepsilon \|\beta(z) w^{-n}\|_{W^{-1}(U_1)} \left( \int_{V_1} e^{-\lambda_n(z)} dV(z) \right)^{1/2}. \end{aligned}$$

Then by Lemma 2 there exists  $C > 0$  such that

$$\|\beta(z) w^{-n}\|_{W^{-1}(U_1)} \leq \frac{C}{n} \|\beta(z) w^{-n}\|_{U_1} = \frac{C}{n} \|\beta\|_{L^2(D(z_0, a), \lambda_n)}.$$

We note that to get the equality above we used the fact that  $\beta$  is supported in  $D(z_0, a)$ . Hence we get

$$\|g_n\|_{L^2(D(z_0, a), \lambda_n)}^2 \leq \left( \varepsilon + \frac{CC_\varepsilon}{n} \right) \|\beta\|_{L^2(D(z_0, a), \lambda_n)} \|1\|_{L^2(V_1, \lambda_n)}.$$

For any  $\varepsilon > 0$  there exists an integer  $n_\varepsilon$  such that

$$\frac{CC_\varepsilon}{n} + \frac{\pi a_1}{\sqrt{n-1}} \leq \varepsilon$$

for  $n \geq n_\varepsilon$ . Then

$$\|g_n\|_{L^2(D(z_0, a), \lambda_n)}^2 \leq 2\varepsilon \|\beta\|_{L^2(D(z_0, a), \lambda_n)} \|1\|_{L^2(V_1, \lambda_n)} \leq 2\varepsilon^2 \|\beta\|_{L^2(D(z_0, a), \lambda_n)}$$

for  $n \geq n_\varepsilon$  because  $U \subset D(z_0, a) \times \{w \in \mathbb{C} : |w| > 1\}$  and

$$\begin{aligned} \|1\|_{L^2(V_1, \lambda_n)} &\leq \|1\|_{L^2(D(z_0, a_1), \lambda_n)} \\ &= \left( \int_{D(z_0, a_1)} \frac{\pi}{n-1} \left( e^{(2n-2)\varphi(z)} - e^{(2n-2)\alpha(z)} \right) dV(z) \right)^{1/2} \\ &\leq \left( \int_{D(z_0, a_1)} \frac{\pi}{n-1} dV(z) \right)^{1/2} \\ &= \frac{\pi a_1}{\sqrt{n-1}}. \end{aligned}$$

Let  $u \in C_0^\infty(D(z_0, a))$  and  $n \geq n_\varepsilon$ . Then

$$\begin{aligned} \int_{D(z_0, a)} |u(z)|^2 e^{\lambda_n(z)} dV(z) &= \sup \left\{ |\langle u, \beta \rangle_{D(z_0, a)}|^2 : \beta \in C_0^\infty(D(z_0, a)), \|\beta\|_{L^2(D(z_0, a), \lambda_n)}^2 \leq 1 \right\} \\ &\leq \sup \left\{ |\langle u, (g_n)_{\bar{z}} \rangle_{D(z_0, a)}|^2 : \|g_n\|_{L^2(D(z_0, a), \lambda_n)}^2 \leq 2\varepsilon^2 \right\} \\ &= \sup \left\{ |\langle u_z, g_n \rangle_{D(z_0, a)}|^2 : \|g_n\|_{L^2(D(z_0, a), \lambda_n)}^2 \leq 2\varepsilon^2 \right\} \\ &\leq 2\varepsilon^2 \int_{D(z_0, a)} |u_z(z)|^2 e^{\lambda_n(z)} dV(z). \end{aligned}$$

There exists  $0 < c < 1$  such that  $e^{-\varphi(z)} < ce^{-\alpha(z)}$  for  $z \in D(z_0, a)$ . Then

$$\frac{\pi}{n-1} e^{(2n-2)\varphi(z)} (1 - c^{2n-2}) < e^{-\lambda_n(z)} < \frac{\pi}{n-1} e^{(2n-2)\varphi(z)}.$$

So for large  $n$  we have

$$\frac{\pi}{2(n-1)} e^{(2n-2)\varphi(z)} < e^{-\lambda_n(z)} < \frac{\pi}{n-1} e^{(2n-2)\varphi(z)}$$

and

$$\begin{aligned} \frac{n-1}{\pi} \int_{D(z_0, a)} |u(z)|^2 e^{(2-2n)\varphi(z)} dV(z) &< \int_{D(z_0, a)} |u(z)|^2 e^{\lambda_n(z)} dV(z) \\ &\leq 2\varepsilon^2 \int_{D(z_0, a)} |u_z(z)|^2 e^{\lambda_n(z)} dV(z) \\ &\leq \frac{4\varepsilon^2(n-1)}{\pi} \int_{D(z_0, a)} |u_z(z)|^2 e^{(2-2n)\varphi(z)} dV(z). \end{aligned}$$

That is, for any  $\varepsilon > 0$  and  $u \in C_0^\infty(D(z_0, a))$

$$(2) \quad \int_{D(z_0, a)} |u(z)|^2 e^{(2-2n)\varphi(z)} dV(z) \leq 4\varepsilon^2 \int_{D(z_0, a)} |u_z(z)|^2 e^{(2-2n)\varphi(z)} dV(z)$$

for large  $n$ .

The estimate in (2) is identical to the one in [CF05, pg. 38, proof of Lemma 10.2]. That is  $\lambda_{n\varphi}^m(D(z_0, a)) \rightarrow \infty$  as  $n \rightarrow \infty$  (see [CF05, Definition 2.3]). Since  $\varphi$  is smooth and subharmonic, [CF05, Theorem 1.5] implies that  $\lambda_{n\varphi}^e(D(z_0, a)) \rightarrow \infty$  as  $n \rightarrow \infty$ . We note that [CF05, Theorem 1.5] implies that if  $\lambda_{n\varphi}^m(D(z_0, a)) \rightarrow \infty$  as  $n \rightarrow \infty$  then  $\lambda_{n\varphi}^e(D(z_0, a)) \rightarrow \infty$  as  $n \rightarrow \infty$ . This is enough to conclude that  $\Gamma_k$  is  $B$ -regular. This argument is contained in the proof of Proposition 9.1 converse of (1) in [CF05, pg 33]. We repeat the argument here for the convenience of the reader. Let  $V = \{z \in D(z_0, a) : \Delta\varphi(z) > 0\}$  and  $K_0 = \overline{D(z_0, a/2)} \setminus V$ . Then  $V$  is open and  $K_0$  is a compact subset of  $D(z_0, a)$ . Furthermore,  $\Delta\varphi = 0$  on  $K_0$ . If  $K_0$  has non-trivial fine interior then it supports a nonzero function  $f \in W^1(\mathbb{C})$  (see [Str10, Proposition 4.17]). Then

$$\lambda_{n\varphi}^e(D(z_0, a)) \leq \frac{\|\nabla f\|^2}{\|f\|^2} < \infty \text{ for all } n.$$



Which is a contradiction. Hence  $K_0$  has empty fine interior which implies that  $K_0$  satisfies property (P) (see [Str10, Proposition 4.17] or [Sib87, Proposition 1.11]). Therefore, for  $M > 0$  there exists an open neighborhood  $O_M$  of  $K_0$  and  $b_M \in C_0^\infty(O_M)$  such that  $|b_M| \leq 1/2$  on  $O_M$  and  $\Delta b_M > M$  on  $K_0$ . Furthermore, using the assumption that  $|w| > 0$  on  $\Gamma_k$  one can choose  $M_1$  such that the function  $g_{M_1}(z, w) = M_1(|w|^2 e^{\varphi(z)} - 1) + b_M(z)$  has the following properties:  $|g_{M_1}| \leq 1$  and the complex Hessian  $H_{g_{M_1}}(W) \geq M\|W\|^2$  on  $\Gamma_k \cap \overline{D(z_0, a)}$  where  $W$  is complex tangential direction. Then [Ayy14, Proposition 3.1.7] implies that  $\Gamma_k \cap \overline{D(z_0, a/2)}$  satisfies property (P) (hence it is  $B$ -regular). Therefore, [Str10, Corollary 4.13] implies that  $\Gamma_k$  is  $B$ -regular.

The computations in the second case (that is  $\rho_{|w|}(z_0, |w_0|) > 0$ ) are very similar. So we will just highlight the differences between the two cases. We define

$$U^{a_1, b_1} = \left\{ (z, w) \in \mathbb{C}^2 : |w| > b_1 |z - z_0|^2 + a_1 \right\}$$

and

$$U_1 = \Omega \cap U^{a_1, b_1} = \left\{ (z, w) \in \mathbb{C}^2 : z \in V_1, e^{-\alpha(z)} < |w| < e^{-\varphi(z)} \right\}$$

where  $V_1$  is a domain in  $\mathbb{C}$  and where  $\alpha(z) = -\log(b_1 |z - z_0|^2 + a_1)$  is a strictly superharmonic function. One can show that  $bU^{a_1, b_1}$  is strongly pseudoconvex. We choose  $a, a_1, b, b_1 > 0$  such that such that  $\overline{D(z_0, a)} \subset V_1$  and  $U$  is given by

$$U = \Omega \cap U_{a,b} = \left\{ (z, w) \in \mathbb{C}^2 : z \in D(z_0, a), e^{-\alpha(z)} < |w| < e^{-\varphi(z)} \right\}$$

where  $U_{a,b} = D(z_0, a) \times \{w \in \mathbb{C} : |w_0| - b < |w| < |w_0| + b\}$ . Furthermore, we define

$$\lambda_n(z) = -\log \left( \frac{\pi}{n+1} \left( e^{-(2n+2)\varphi(z)} - e^{-(2n+2)\alpha(z)} \right) \right)$$

for  $n = 0, 1, 2, \dots$  and by scaling  $U_1$  in  $w$  variable if necessary, we will assume that  $U_1 \subset D(z_0, a_1) \times \{w \in \mathbb{C} : |w| < 1\}$  so that  $\|1\|_{L^2(D(z_0, a_1), \lambda_n)}$  goes to zero as  $n \rightarrow \infty$ . One can check that  $\lambda_n$  is subharmonic for all  $n$ .

We take functions  $\beta \in C_0^\infty(D(z_0, a))$  and consider symbols  $\psi \in C^\infty(\overline{V_1})$  such that  $\psi_{\bar{z}} = \beta$ . Then we consider the functions  $H_\psi w^n$  for  $n = 0, 1, 2, \dots$ . Calculations similar to the ones in the previous case reveal that  $g_n(z)w^n = H_\psi w^n$  where  $g_n = S_{\lambda_n}^{V_1}(\beta d\bar{z})$ . Using similar manipulations and again the compactness estimate (1) we conclude that for any  $\varepsilon > 0$  there exists an integer  $n_\varepsilon$  such that for  $u \in C_0^\infty(D(z_0, a))$  and  $n \geq n_\varepsilon$  we have

$$(3) \quad \int_{D(z_0, a)} |u(z)|^2 e^{(2n+2)\varphi(z)} dV(z) \leq \varepsilon \int_{D(z_0, a)} |u_z(z)|^2 e^{(2n+2)\varphi(z)} dV(z).$$

Finally, an argument similar to the one right after (2) implies that  $\Gamma_k$  is  $B$ -regular.  $\square$

#### ACKNOWLEDGEMENT

We would like to thank the anonymous referee for constructive comments.

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